

Propositional Logic

Question: How do we formalize the definitions and reasoning we use in our proofs?

Where We're Going

- ***Propositional Logic*** (Today)
 - Reasoning about Boolean values.
- ***First-Order Logic*** (Wednesday/Friday)
 - Reasoning about properties of multiple objects.

Outline for Today

- ***Propositional Variables***
 - Booleans, math edition!
- ***Propositional Connectives***
 - Linking things together.
- ***Truth Tables***
 - Rigorously defining connectives.
- ***Simplifying Negations***
 - Mechanically computing negations.

Propositional Logic

TakeMath51 \vee *TakeCME100*

\neg *FirstSucceed* \rightarrow *TryAgain*

IsCardinal \wedge *IsWhite*

TakeMath51 ∨ TakeCME100

¬FirstSucceed → TryAgain

IsCardinal ∧ IsWhite

TakeMath51 \vee *TakeCME100*

\neg *FirstSucceed* \rightarrow *TryAgain*

IsCardinal \wedge *IsWhite*

These are **propositional variables**. Each propositional variable stands for a **proposition**, something that is either true or false.

TakeMath51 \vee *TakeCME100*

\neg *FirstSucceed* \rightarrow *TryAgain*

IsCardinal \wedge *IsWhite*

These are **propositional connectives**, which link propositions into larger propositions

Propositional Variables

- In propositional logic, individual propositions are represented by ***propositional variables***.
- In a move that contravenes programming style conventions, propositional variables are usually represented as lower-case letters, such as *p*, *q*, *r*, *s*, etc.
 - That said, there's nothing stopping you from using multiletter names!
- Each variable can take one one of two values: true or false. You can think of them as **bool** values.

Propositional Connectives

- There are seven propositional connectives, five of which will be familiar from programming.
- First, there's the logical "NOT" operation:

$\neg p$

- You'd read this out loud as "not p ."
- The fancy name for this operation is ***logical negation***.

Propositional Connectives

- There are seven propositional connectives, five of which will be familiar from programming.
- Next, there's the logical "AND" operation:

$$p \wedge q$$

- You'd read this out loud as " p and q ."
- The fancy name for this operation is ***logical conjunction***.

Propositional Connectives

- There are seven propositional connectives, five of which will be familiar from programming.
- Then, there's the logical "OR" operation:

$$p \vee q$$

- You'd read this out loud as "*p* or *q*."
- The fancy name for this operation is **logical disjunction**. This is an *inclusive* or.

Propositional Connectives

- There are seven propositional connectives, five of which will be familiar from programming.
- There's also the "truth" connective:

T

- You'd read this out loud as "true."
- Although this is technically considered a connective, it "connects" zero things and behaves like a variable that's always true.

Propositional Connectives

- There are seven propositional connectives, five of which will be familiar from programming.
- Finally, there's the “false” connective.

⊥

- You'd read this out loud as “false.”
- Like \top , this is technically a connective, but acts like a variable that's always false.

Truth Tables

- A ***truth table*** is a table showing the truth value of a propositional logic formula as a function of its inputs.
- Let's go look at the truth tables for the connectives we've seen so far:

\neg \wedge \vee \top \perp

Summary of Important Points

- The \vee connective is an *inclusive* “or.” It's true if at least one of the operands is true.
 - Similar to the `||` operator in C, C++, Java, etc. and the `or` operator in Python.
- If we need an exclusive “or” operator, we can build it out of what we already have.
- Try this yourself! Take a minute to combine these operators together to form an expression that represents the exclusive or of p and q (something that's true if and only if exactly one of p and q are true.)

Quick Question:

What would I have to show you to convince you that the statement $p \wedge q$ is false?

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What would I have to show you to convince you that the statement $p \vee q$ is false?

de Morgan's Laws

$\neg(p \wedge q)$ *is equivalent to* $\neg p \vee \neg q$

$\neg(p \vee q)$ *is equivalent to* $\neg p \wedge \neg q$

de Morgan's Laws in Code

- **Pro tip:** Don't write this:

```
if (!(p() && q())) {  
    /* ... */  
}
```

- Write this instead:

```
if (!p() || !q()) {  
    /* ... */  
}
```

- (This even short-circuits correctly: if `p()` returns false, `q()` is never evaluated.)

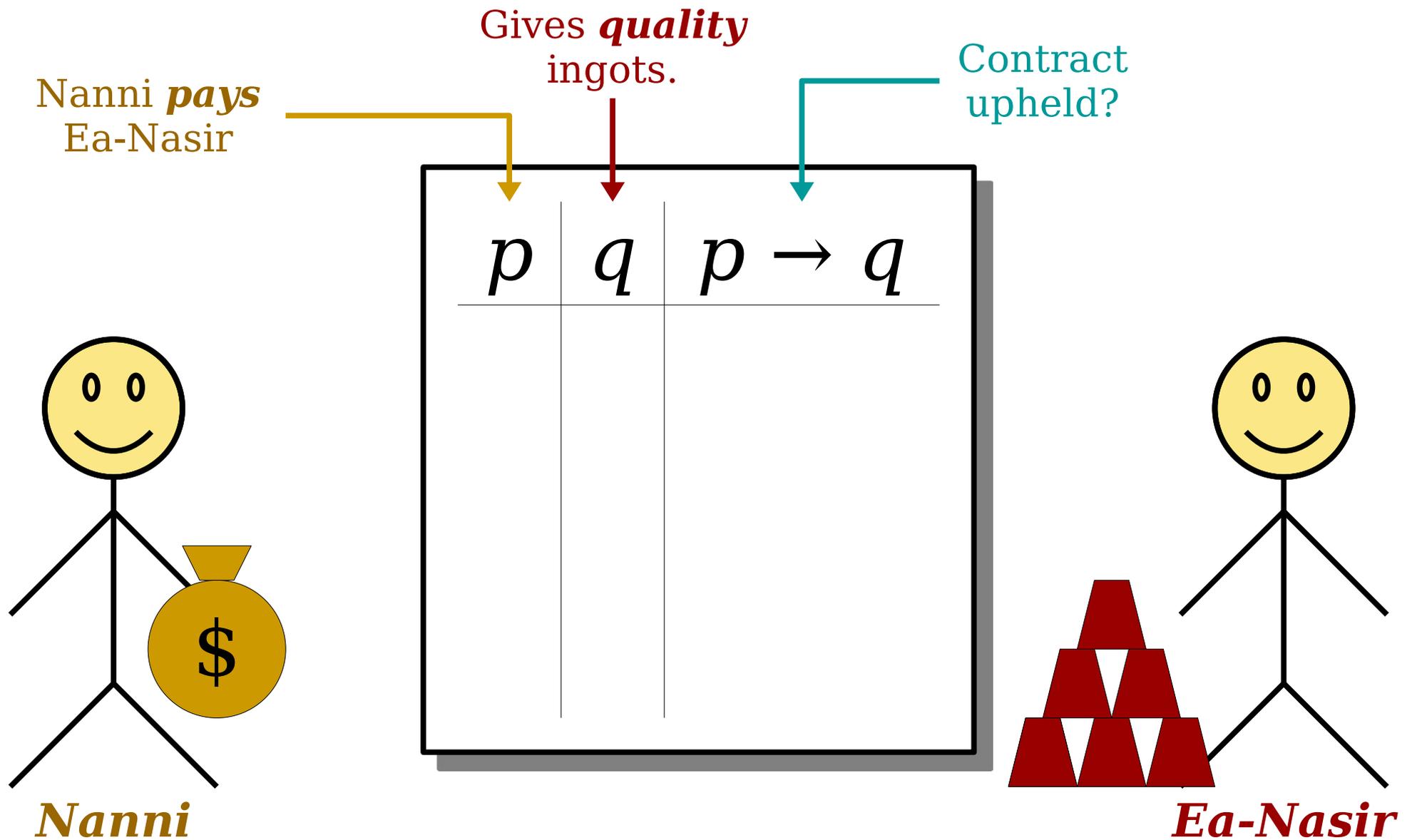
Mathematical Implication

Implication

- We can represent implications using this connective:

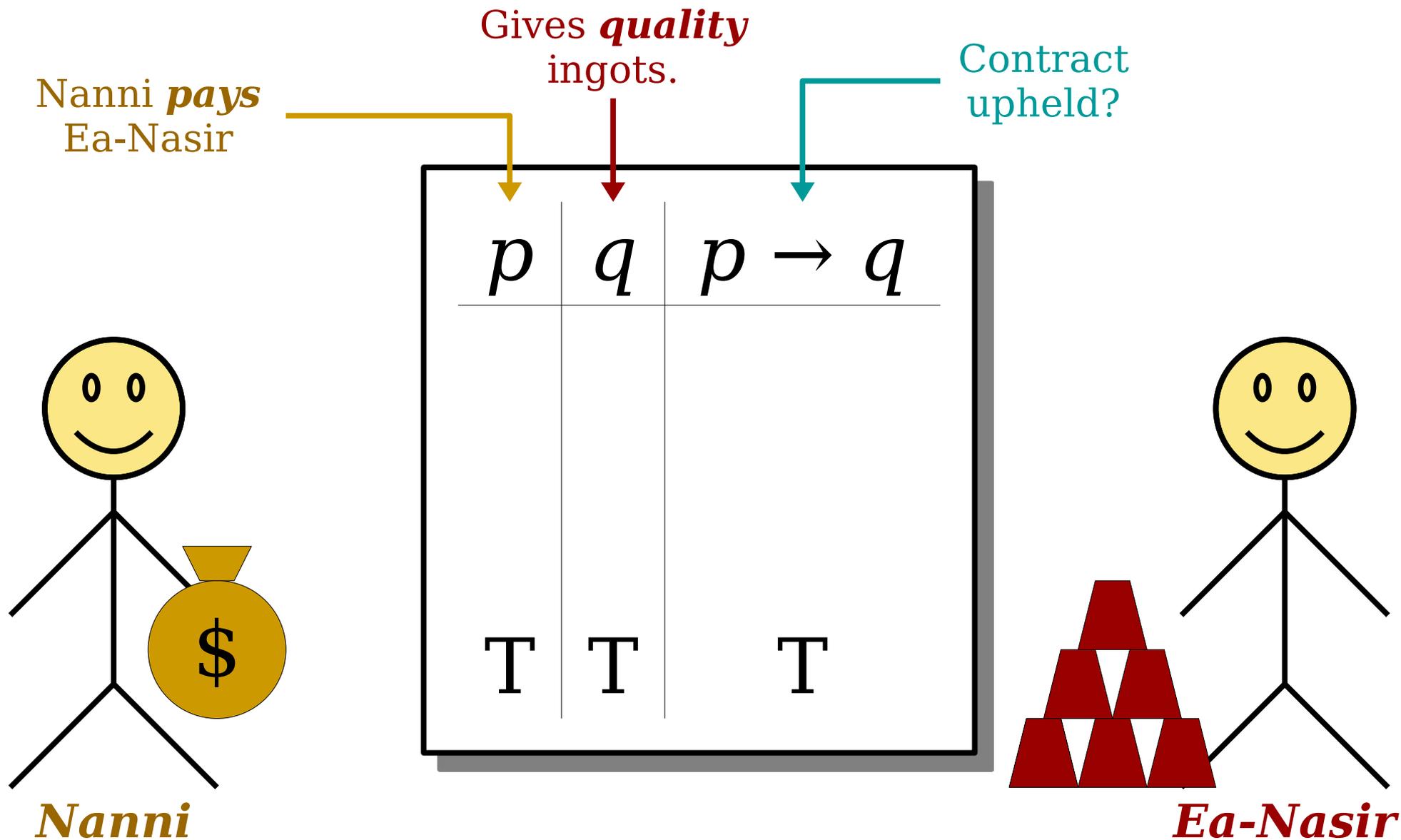
$$p \rightarrow q$$

- You'd read this out loud as “ p implies q .”
 - The fancy name for this is the ***material conditional***.
- ***Question:*** What should the truth table for $p \rightarrow q$ look like?
- Pull out a sheet of paper, make a guess, and talk things over with your neighbors!



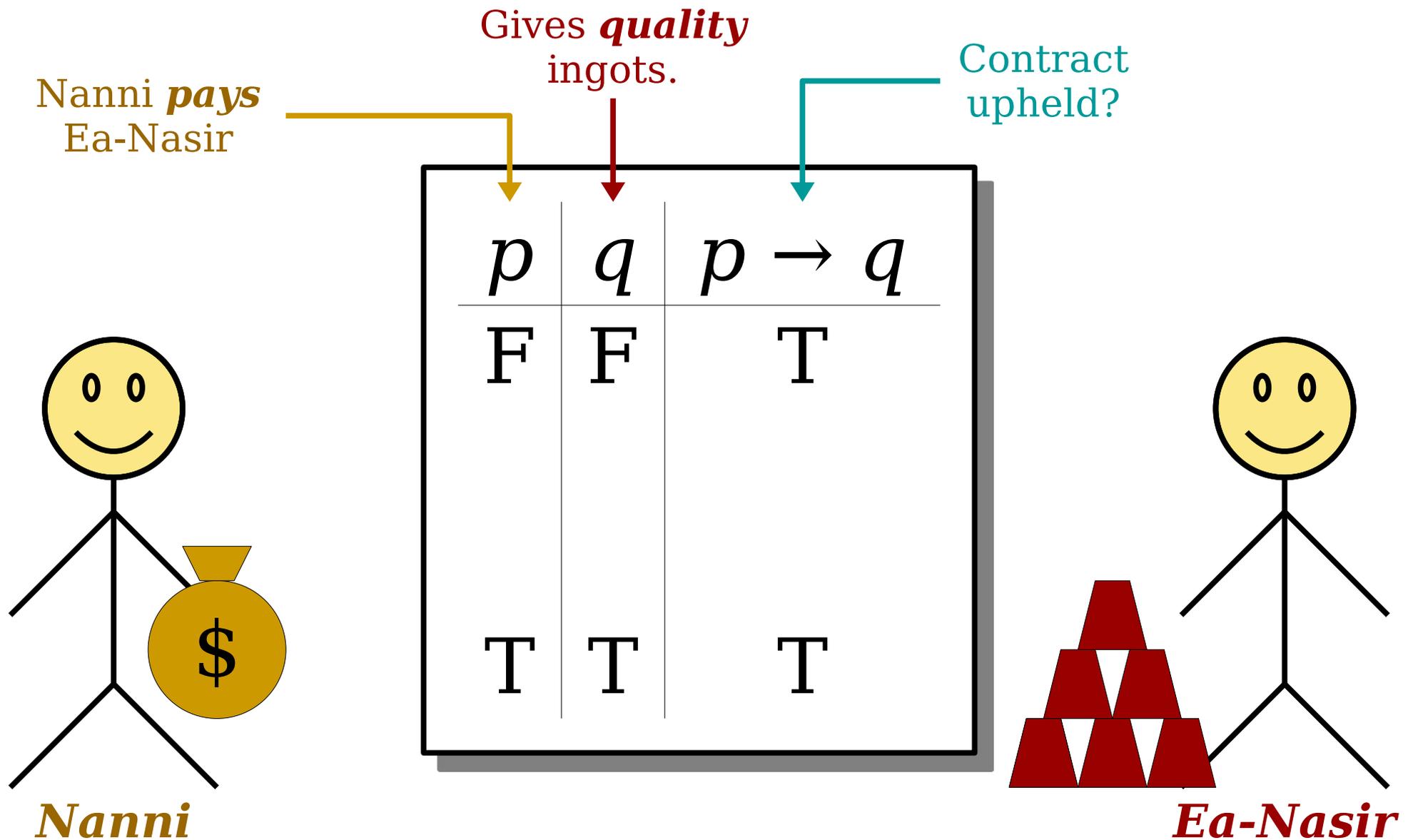
Ancient Contract:

If Nanni pays money to Ea-Nasir, then Ea-Nasir will give Nanni quality copper ingots.



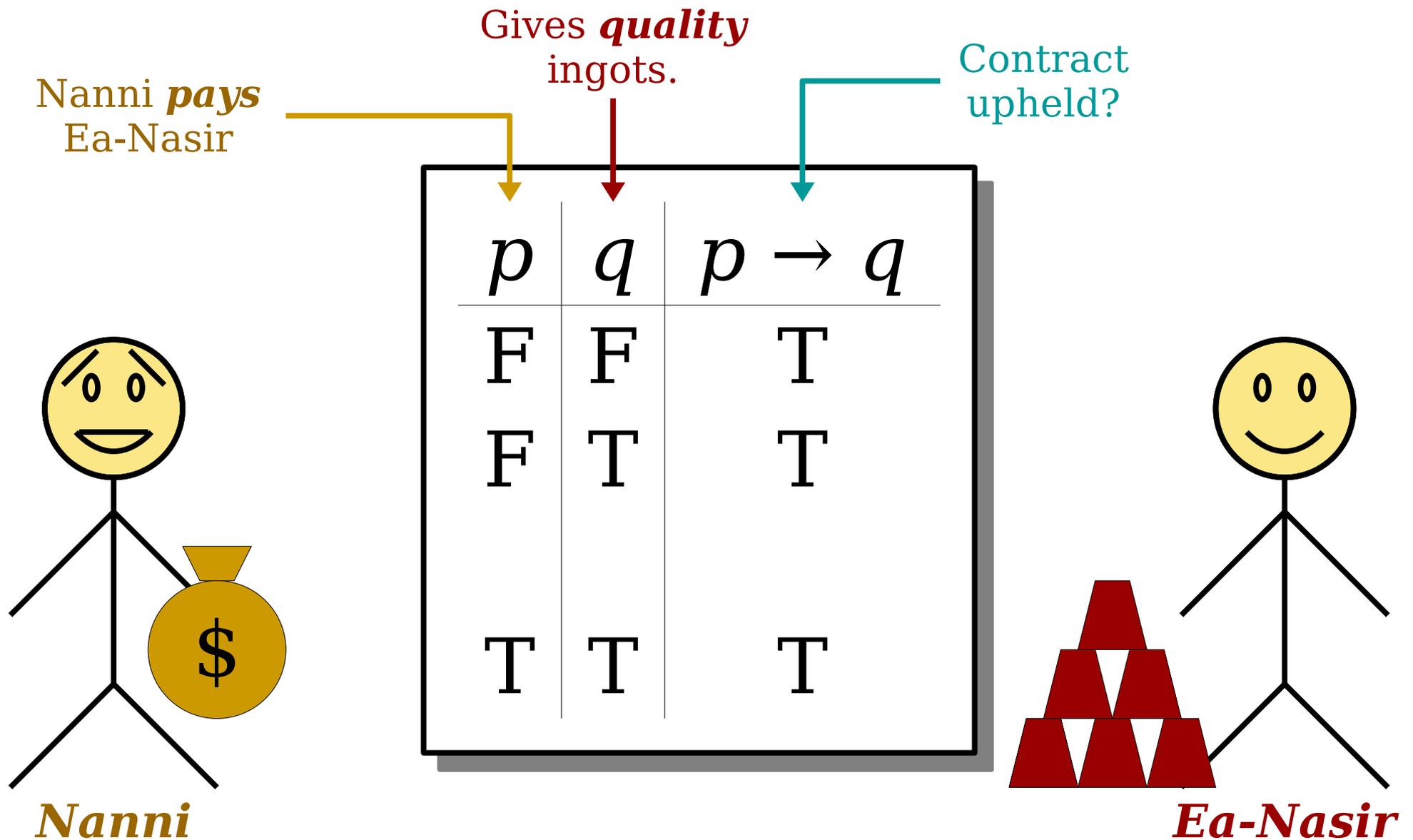
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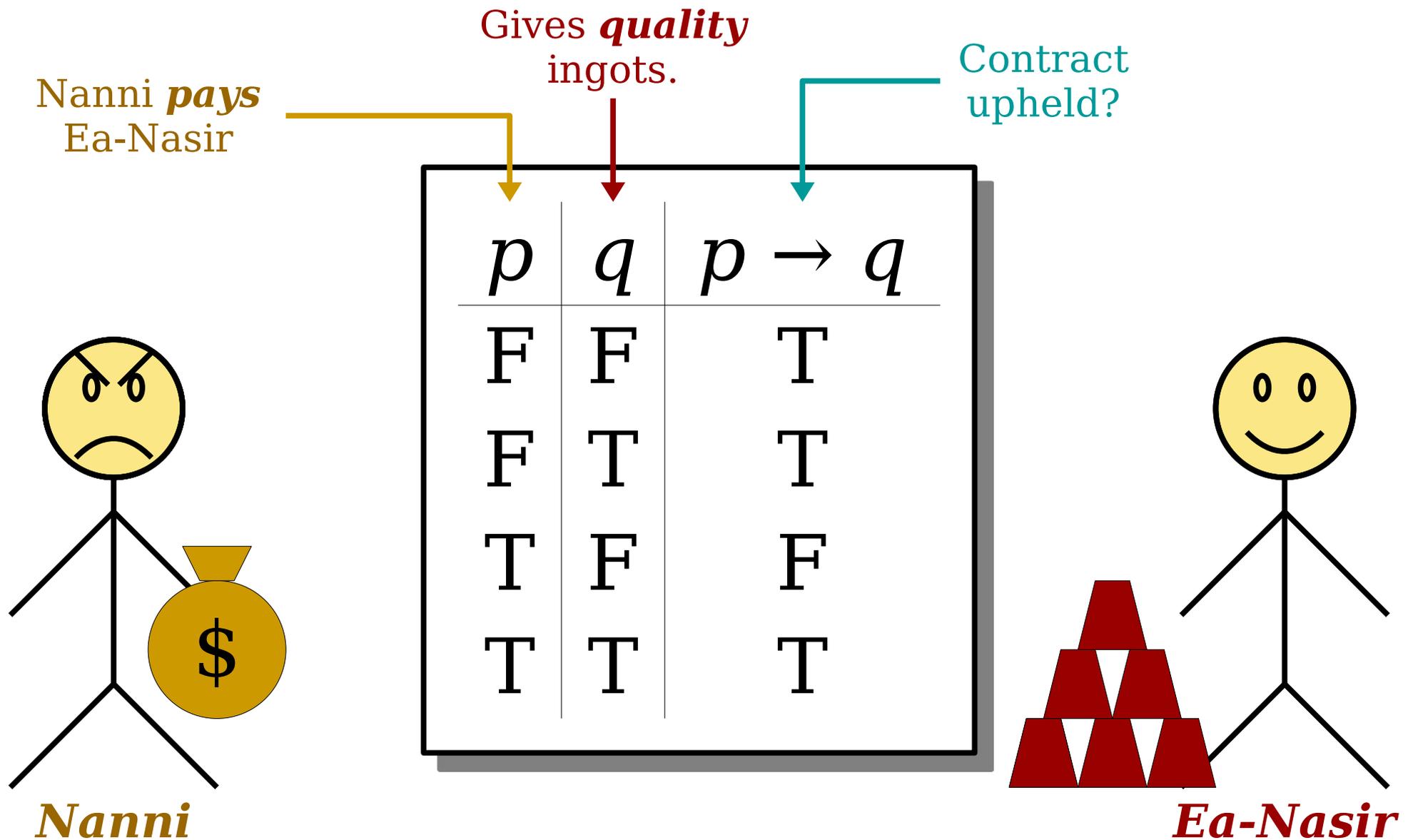
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p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

An implication is false only when the antecedent is true and the consequent is false.

Every formula is either true or false, so these other entries have to be true.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Important observation:

The statement $p \rightarrow q$ is true whenever $p \wedge \neg q$ is false.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

An implication with a false antecedent is called ***vacuously true***.

An implication with a true consequent is called ***trivially true***.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Please commit this table to memory. We're going to need it, extensively, over the next couple of weeks.

Fun Fact 1: The Contrapositive Revisited

Fun Fact 2: Proof by Contradiction

Fun Fact 3: Implies, Another Way

An Important Equivalence

- Earlier, we talked about the truth table for $p \rightarrow q$. We chose it so that

$p \rightarrow q$ is equivalent to $\neg(p \wedge \neg q)$

- Later on, this equivalence will be incredibly useful:

$\neg(p \rightarrow q)$ is equivalent to $p \wedge \neg q$

Another Important Equivalence

- Here's a useful equivalence. Start with

$$**p \rightarrow q** \text{ is equivalent to } \neg(\mathbf{p \wedge \neg q})$$

- By de Morgan's laws:

$$**p \rightarrow q** \text{ is equivalent to } \neg(\mathbf{p \wedge \neg q})$$

$$\text{is equivalent to } \neg\mathbf{p \vee \neg\neg q}$$

$$\text{is equivalent to } \neg\mathbf{p \vee q}$$

- Thus **p \rightarrow q** is equivalent to **\neg p \vee q**

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$p \rightarrow q$ is equivalent to $\neg(p \wedge \neg q)$

- By de Morgan's laws:

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- Thus $p \rightarrow q$ is equivalent to $\neg p \vee q$

If p is false, then $\neg p \vee q$ is true. If p is true, then q has to be true for the whole expression to be true.

The Biconditional Connective

The Biconditional Connective

- On Friday, we saw that “ p if and only if q ” means both that $p \rightarrow q$ and $q \rightarrow p$.
- We can write this in propositional logic using the ***biconditional*** connective:

$$p \leftrightarrow q$$

- This connective’s truth table has the same meaning as “ p implies q and q implies p .”
- Based on that, what should its truth table look like?
- Take a guess, and talk it over with your neighbor!

Biconditionals

- The ***biconditional*** connective $p \leftrightarrow q$ is read “ p if and only if q .”
- Here's its truth table:

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Biconditionals

- The ***biconditional*** connective $p \leftrightarrow q$ is read “ p if and only if q .”
- Here's its truth table:

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

One interpretation of \leftrightarrow is to think of it as equality: the two propositions must have equal truth values.

Negating a Biconditional

- How do we simplify

$$\neg(p \leftrightarrow q)$$

using the tools we've seen so far?

- There are many options, but here are our two favorites:

$$p \leftrightarrow \neg q$$

$$\neg p \leftrightarrow q$$

Operator Precedence

- How do we parse this statement?

$$\neg x \rightarrow y \vee z \rightarrow x \vee y \wedge z$$

- Operator precedence for propositional logic:

\neg

\wedge

\vee

\rightarrow

\leftrightarrow

- All operators are right-associative.
- We can use parentheses to disambiguate.

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$$(\neg x) \rightarrow y \vee z \rightarrow x \vee y \wedge z$$

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$$(\neg x) \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$$

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Operator Precedence

- The main points to remember:
 - \neg binds to whatever immediately follows it.
 - \wedge and \vee bind more tightly than \rightarrow .
- We will commonly write expressions like $p \wedge q \rightarrow r$ without adding parentheses.
- For more complex expressions, we'll try to add parentheses.
- Confused? ***Please ask!***

The Big Table

Connective	Read Aloud As	C++ Version	Fancy Name
\neg	“not”	!	Negation
\wedge	“and”	&&	Conjunction
\vee	“or”		Disjunction
\top	“true”	true	Truth
\perp	“false”	false	Falsity
\rightarrow	“implies”	<i>see PS2!</i>	Implication
\leftrightarrow	“if and only if”	<i>see PS2!</i>	Biconditional

Time-Out for Announcements!

Ask Me Anything

- Between lectures, I'll make a post on EdStem soliciting your questions.
- This is a chance to ask questions about any topic: course material, life advice, music recommendations, etc.
 - Just keep things civil and only ask questions if you really want to hear the answer.
- Feel free to ask questions you like. I'll take some of the more popular questions in lecture.

Office Hours

- Office hours start today. Think of them as “drop-in help hours” where you can ask questions on problem sets, lecture topics, etc.
 - Check the Guide to Office Hours on the course website for the schedule.
- Most office hours are held in person in the Huang basement. A few are purely online. Mine are in my office, Durand 317.
- Once you arrive, sign up on QueueStatus so that we can help people in the order they arrived:

<https://questatus.com/queues/782>
- Office hours are much less crowded earlier in the week than later.

Back to CS103!

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are
 - Negation: $\neg p$
 - Conjunction: $p \wedge q$
 - Disjunction: $p \vee q$
 - Truth: \top
 - Falsity: \perp
 - Implication: $p \rightarrow q$
 - Biconditional: $p \leftrightarrow q$

Negation Practice

- Here's a propositional formula that contains some negations. Simplify it as much as possible:

$$\neg(p \wedge q \rightarrow r \vee s)$$

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“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$ ”

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$$x + y = 16 \rightarrow x \geq 8 \vee y \geq 8$$

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“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$ ”

$$x < 8 \wedge y < 8 \rightarrow x + y \neq 16$$

“If $x < 8$ and $y < 8$, then $x + y \neq 16$ ”

Theorem: If $x + y = 16$, then $x \geq 8$ or $y \geq 8$.

Proof: We will prove the contrapositive, namely, that if $x < 8$ and $y < 8$, then $x + y \neq 16$.

Pick x and y where $x < 8$ and $y < 8$. We want to show that $x + y \neq 16$. To see this, note that

$$\begin{aligned}x + y &< 8 + y \\ &< 8 + 8 \\ &= 16.\end{aligned}$$

This means that $x + y < 16$, so $x + y \neq 16$, which is what we needed to show. ■

Why This Matters

- Propositional logic is a tool for reasoning about how various statements affect one another.
- To better understand how to prove a result, it often helps to translate what you're trying to prove into propositional logic first.
- That said, propositional logic isn't expressive enough to capture all statements. For that, we need something more powerful.

Next Time

- ***First-Order Logic***
 - Reasoning about groups of objects.
- ***First-Order Translations***
 - Expressing yourself in symbolic math!